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LETTER TO THE EDITOR

Photon statistics of optical frequency up-conversion with stochastic pumping

S Kryszewski and J Chrostowski

Institute of Experimental Physics, University of Warsaw, Hoża 69, 00-681 Warsaw, Poland

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Abstract. The frequency up-conversion process with stochastic coupling function is considered. It is shown that the total photon number increases with time. The antibunching effect ($g^{(2)} < 1$) is predicted regardless of the mean photon numbers available in the process.

Parameteric frequency conversion has recently received much attention. The effective Hamiltonian describing that phenomenon has the form

$$H = \hbar\omega_a a^\dagger a + \hbar\omega_b b^\dagger b + \hbar\{g(t)\exp[\mathrm{i}(\omega_a - \omega_b)t]ab^\dagger + \mathrm{HC}\} \tag{1}$$

where ω_a, $a^\dagger$, a, ω_b, $b^\dagger$, b, denote the frequencies, creation and annihilation operators for modes A and B respectively. It is worth noting that in frequency conversion three electromagnetic modes are coupled. However, in the derivation of Hamiltonian (1) one uses the well known parametric approximation (Tucker and Walls 1969b), thus reducing the problem to the two-mode interaction. In Tucker and Walls (1969a, b) the coupling function $g(t)$ is taken to be constant. Lu (1973) assumes $g(t)$ to be the ordinary time function. Crosignani *et al* (1971) consider $g(t)$ to be a stochastic process. They construct the Fokker–Planck equation for the P-representation of the appropriate density operators and find its statistical moments.

In this Letter we assume that the coupling between modes A and B is described by the stochastic function of time $g(t)$. We adopt an entirely different approach which seems to be more general than that presented by Crosignani *et al*, because we can treat arbitrary quantum states which might not possess the P-representation.

The stochastic character of the function $g(t)$ may have two sources: (i) the fluctuations of the classically treated amplitude of the pumping mode; (ii) the fluctuations of the non-linear polarisability of the medium in which the frequency conversion occurs. We assume that the function $g(t)$ describes small fluctuations around its (time-independent) mean value $\bar{g}$, i.e. $g(t) = \bar{g} + \epsilon g_1(t)$, where ϵ is a small parameter and $g_1(t)$ is a stochastic stationary Gaussian process with mean value and correlation function: $\langle g_1(t)\rangle_s = 0$; $\langle g_1^*(t)g_1(t')\rangle_s = 2D\delta(t-t')$ respectively, and with other second-order moments vanishing.

The Hamiltonian (1) does not include damping effects, therefore we add the phenomenological damping terms to the Heisenberg equations of motion for the annihilation operators. Thus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}a(t)=-\mathrm{i}\omega_a a(t)-\tfrac{1}{2}\Gamma a(t)-\mathrm{i}g^*(t)\exp[-\mathrm{i}(\omega_a-\omega_b)t]b(t) \tag{2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}b(t)=-\mathrm{i}\omega_b b(t)-\tfrac{1}{2}\Gamma b(t)-\mathrm{i}g(t)\exp[\mathrm{i}(\omega_a-\omega_b)t]a(t). \tag{2b}$$

For simplicity we assume the same damping constants for both modes. One should note that our considerations are valid for times $t \ll 1/\Gamma$, so we take damping to be small. Equations (2) are coupled and the stochastic character of the function $g(t)$ does not allow us to find the exact solution. Therefore we adopt the perturbative approach looking for the solution of the form

$$a(t)=a_{(0)}(t)+\epsilon a_1(t) \tag{3a}$$

$$b(t)=b_{(0)}(t)+\epsilon b_1(t) \tag{3b}$$

i.e. we retain only terms in the first order in the parameter ϵ. The unperturbed solution obtained by putting $g(t)=\bar{g}$ in equations (3) reads

$$a_{(0)}(t)=\exp(-\mathrm{i}\omega_a t-\tfrac{1}{2}\Gamma t)(a_0\cos\kappa t+\mathrm{e}^{-\mathrm{i}\psi}b_0\sin\kappa t) \tag{4a}$$

$$b_{(0)}(t)=\exp(-\mathrm{i}\omega_b t-\tfrac{1}{2}\Gamma t)(b_0\cos\kappa t-\mathrm{e}^{+\mathrm{i}\psi}a_0\sin\kappa t) \tag{4b}$$

where a_0 and b_0 are the initial annihilation operators for modes A and B, and where $\kappa=|\bar{g}|$, $\mathrm{e}^{\mathrm{i}\psi}=\mathrm{i}\bar{g}/\kappa$. The first-order corrections are given by

$$a_1(t)=\int_0^t \mathrm{d}t'[-\mathrm{i}g_1(t')\,\mathrm{e}^{-\mathrm{i}\psi}\sin[\kappa(t-t')]\exp[-\mathrm{i}\omega_a(t-t')]a_{(0)}(t')$$
$$-\mathrm{i}g_1^*(t')\cos[\kappa(t-t')]\exp[-\mathrm{i}(\omega_a t-\omega_b t')]b_{(0)}(t')] \tag{5a}$$

$$b_1(t)=\int_0^t \mathrm{d}t'[\mathrm{i}g_1^*(t')\,\mathrm{e}^{\mathrm{i}\psi}\sin[\kappa(t-t')]\exp[-\mathrm{i}\omega_b(t-t')]b_{(0)}(t')$$
$$-\mathrm{i}g_1(t')\cos[\kappa(t-t')]\exp[-\mathrm{i}(\omega_b t-\omega_a t')]a_{(0)}(t')]. \tag{5b}$$

It is necessary to emphasise that the corrections are not damped because we neglected the small damping of the small corrections. The constant κ gives us the time scale of the process under consideration; we assume that damping and stochastic perturbation are small, i.e. $2D\epsilon^2/\kappa$, $\Gamma/\kappa \ll 1$. These assumptions allow us to take only the first-order approximation (3) and neglect the damping of the corrections. Equations (3)–(5) give the time evolution of the amplitudes of the modes A and B.

In order to study the statistical properties of modes A and B we have to specify the initial state of the system. We assume modes A and B to be initially independent and described by the density operator

$$\rho(0)=\rho_B(0)|0\rangle_A\,{}_A\langle 0|$$

where $|0\rangle_A$ is the vacuum state for mode A and $\rho_B(0)$ is an arbitrary density operator for mode B. Thus mode A is generated during the process and we focus our attention mainly on its properties.

The calculation of the first-order coherence functions is rather lengthy but straightforward. Using (3)–(5) one gets (for $t_1 > t_2$)

$$G_A^{(1)}(t_1, t_2) = \langle\langle a^\dagger(t_1)a(t_2)\rangle\rangle_s$$
$$= \mathring{N}_b \exp[i\omega_a(t_1 - t_2)]\{\exp[-\tfrac{1}{2}\Gamma(t_1+t_2)] \sin \kappa t_1 \sin \kappa t_2 + D\epsilon^2/\Gamma(f_+(t_1, t_2, t_2) + J(t_1, t_2, t_2))\} \quad (6)$$

$$G_B^{(1)}(t_1, t_2) = \langle\langle b^\dagger(t_1)b(t_2)\rangle\rangle_s$$
$$= \mathring{N}_b \exp[i\omega_b(t_1 - t_2)]\{\exp[-\tfrac{1}{2}\Gamma(t_1+t_2)] \cos \kappa t_1 \cos \kappa t_2 + D\epsilon^2/\Gamma(f_-(t_1, t_2, t_2) - J(t_1, t_2, t_2))\} \quad (7)$$

where $\mathring{N}_b = \mathrm{Tr}(\rho_B(0)b_0^\dagger b_0)$ is the initial number of photons in the mode B, and

$$f_\pm(t_1, t_2, t_3) = (1 - e^{-\Gamma t_3})\{\cos[\kappa(t_1 - t_2)] + \tfrac{1}{2}\cos[\kappa(t_1 + t_2)]\}$$

$$J(t_1, t_2, t_3) = \tfrac{1}{2}\Gamma \int_0^{t_3} dt'\, e^{-\Gamma t'} \cos[\kappa(4t' - t_1 - t_2)].$$

It is easily seen that the coherence functions are the sums of the unperturbed function and some correction due to the stochastic perturbation.

The average photon numbers $\langle\langle N_a(t)\rangle\rangle_s$ and $\langle\langle N_b(t)\rangle\rangle_s$ can be obtained from (6) and (7) respectively by putting $t_1 = t_2 = t$. The unperturbed terms give the well known energy oscillations between the two modes, while the perturbed terms describe the effect of injection of photons from the pump mode. One can show that for $t > 0$ the number of photons in each of the modes is never equal to zero.

The injection of photons is seen most clearly when the total number of photons is considered. From (6), (7) we obtain

$$\langle\langle N_a(t) + N_b(t)\rangle\rangle_s = \mathring{N}_b[2D\epsilon^2/\Gamma - e^{-\Gamma t}(2D\epsilon^2/\Gamma - 1)] \quad (8)$$

and hence $\langle\langle N_a(t) + N_b(t)\rangle\rangle_s$ increases in time provided the condition

$$2D\epsilon^2 > \Gamma \quad (9)$$

is satisfied. Equation (9) states that when the stochastic perturbation dominates over damping then the photon number increases in time (figure 1). When the coupling $g(t)$ = constant, as discussed by Tucker and Walls (1969a, b) the total photon number operator commutes with the Hamiltonian and hence $N_a(t) + N_b(t)$ is the constant of motion. In our case that commutation relation does not hold due to the stochastic character of the coupling function $g(t)$. Equation (8) resembles Brownian motion when Γ tends to zero. Then we have

$$\langle\langle N_a(t) + N_b(t)\rangle\rangle_s \to \mathring{N}_b(1 + 2\epsilon^2 Dt).$$

As in the case of Brownian motion the linear increase in time of the quantities quadratic in the field amplitudes (the analogues of displacement of the Brownian particle) is observed.

Analogous calculations can be performed to obtain the second-order coherence function $G_A^{(2)}(t_1, t_2, t_1, t_2)$ measured in the Hanbury-Brown and Twiss experiment (see

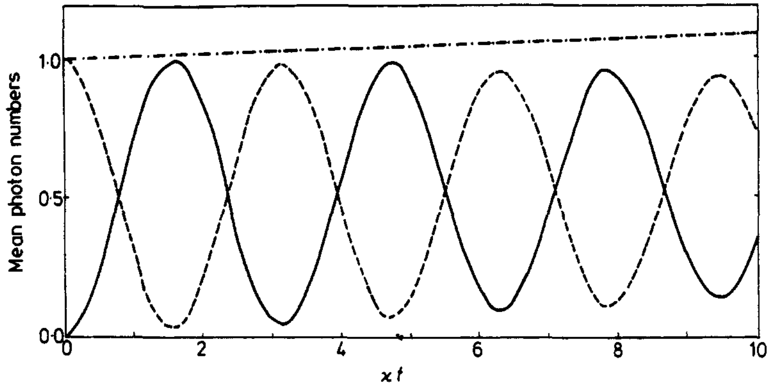

Figure 1. Time dependence of mean photon numbers (normalised to unity) in frequency up-conversion process. The full curve corresponds to the generated mode A, the broken curve to mode B. The chain curve represents the total photon number $\langle\langle N_a(t)+N_b(t)\rangle\rangle_s$. $\Gamma/\kappa=0{\cdot}01$ and $D\epsilon^2/\Gamma=1$.

e.g. Loudon 1973). Then we have (for $t_1>t_2$)

$$
\begin{aligned}
G_A^{(1)}&(t_1, t_2, t_1, t_2)\\
&=\langle\langle a^\dagger(t_1)a^\dagger(t_2)a(t_1)a(t_2)\rangle\rangle_s\\
&=\langle b_0^\dagger b_0^\dagger b_0 b_0\rangle[\mathring{N}_b^{-1}\langle\langle N_a(t_1)\rangle\rangle_s\,\mathrm{e}^{-\Gamma t_2}\sin^2\kappa t_2\\
&\quad+\mathring{N}_b^{-1}\langle\langle N_a(t_2)\rangle\rangle_s\,\mathrm{e}^{-\Gamma t_1}\sin^2\kappa t_1-\mathrm{e}^{-\Gamma(t_1+t_2)}\sin^2\kappa t_1\sin^2\kappa t_2\\
&\quad+(D\epsilon^2/\Gamma)\sin(2\kappa t_1)\sin(2\kappa t_2)\,\mathrm{e}^{-\frac{1}{2}\Gamma(t_1+t_2)}(1-\mathrm{e}^{-\Gamma t_2})].
\end{aligned}
\tag{10}
$$

For $t_1<t_2$ the indices have to be interchanged. The term $\langle b_0^\dagger b_0^\dagger b_0 b_0\rangle$ depends on the initial statistics of the mode B. Formula (10) enables us to find the second-order degree of coherence, defined as

$$g_A^{(2)}(t_1, t_2, t_1, t_2)=|G_A^{(2)}(t_1, t_2, t_1, t_2)|\langle\langle N_a(t_1)\rangle\rangle_s^{-1}\langle\langle N_a(t_2)\rangle\rangle_s^{-1}.$$

Figure 2 shows the dependence of $g_A^{(2)}$ on t_1 with t_2 fixed. For initially coherent mode B $\langle b_0^\dagger b_0^\dagger b_0 b_0\rangle\langle b_0^\dagger b_0\rangle^{-2}=1$ and $g_A^{(2)}(t_1, t_2, t_1, t_2)$ can take on values smaller than unity.

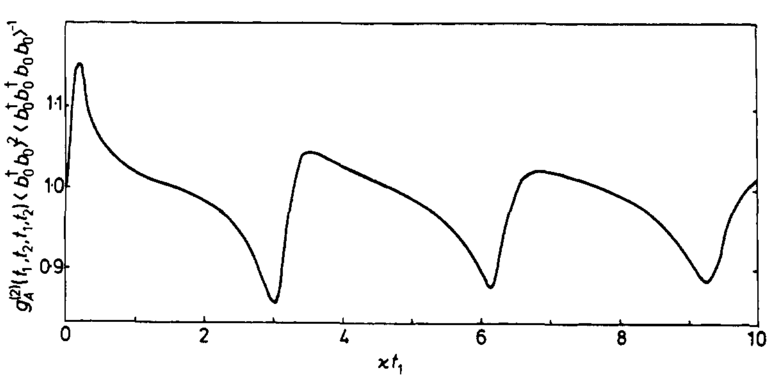

Figure 2. The dependence of $g_A^{(2)}$ on time t_1 while t_2 is fixed. $\Gamma/\kappa=0{\cdot}01$, $D\epsilon^2/\Gamma=1$, $\kappa t_2=0{\cdot}2$.

Hence the effect of antibunching occurs. It is necessary to stress that there is no restriction on the number of photons in any mode, therefore experimental observation of antibunching is not limited to exceedingly weak beams. We wish to emphasise that even when the state of the mode B differs slightly from the coherent state it is still possible to observe antibunching.

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